

Dense Linear Manifolds of Monsters¹

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In this paper the new concept of totally omnipresent operators is introduced. These operators act on the space of holomorphic functions of a domain in the complex plane. The concept is more restrictive than that of strongly omnipresent operators, also introduced by the authors in an earlier work, and both of them are related to the existence of functions whose images under such operators exhibit an extremely wild behaviour near the boundary. Sufficient conditions for an operator to be totally omnipresent as well as several outstanding examples are provided. After extending a statement of the first author about the existence of large linear manifolds of hypercyclic vectors for a sequence of suitable continuous linear mappings, it is shown that there is a dense linear manifold of holomorphic monsters in the sense of Luh, so completing earlier nice results due to Luh and Grosse-Erdmann. © 2002 Elsevier Science (USA)

Key Words: holomorphic monster; T -monster; strongly omnipresent operator; totally omnipresent operator; dense linear manifold; hypercyclic sequence; composition operator; infinite order linear differential operator; integral operator.

1. INTRODUCTION

In 1985 Luh [22] introduced the concept of holomorphic monsters. Roughly speaking, a holomorphic monster in the sense of Luh is a holomorphic function on a simply connected domain G of the complex plane such that it and all its derivatives and antiderivatives possess an extremely wild behaviour near the boundary, see below. Luh proved the existence of a *dense subset* of monsters in the space $H(G)$ of holomorphic functions on G , endowed with the compact-open topology. Note that $H(G)$ is a Fréchet space, hence a Baire space. Two years later, Grosse-Erdmann,

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by using techniques of functional analysis via certain composition–differentiation–antidifferentiation operators, showed that, in fact, there exists a *residual* set of monsters in $H(G)$ [18, Kapitel 3]. In this work we will establish, among other results, the existence of a *dense linear manifold* of holomorphic monsters. Consequently, the set of Luh monsters is large not only topologically but also algebraically. The reader is referred to [23, 24, 27] for further interesting results on this topic.

As a matter of fact, we will state our main result (Theorem 5.1) in a much more general form, by means of the introduction of the notion of totally omnipresent operators, see Section 2. This notion is strictly stronger than that of strongly omnipresent operators, which we recall shortly together with the related concept of T -monsters, both of them introduced by the authors in [6]. In the present paper we strengthen (Theorem 3.1) a recent statement of the first author [4] (see Theorem 1.1) about the existence of large linear manifolds of hypercyclic vectors for a sequence of continuous linear mappings. Theorem 5.1 is extracted as a consequence. Furthermore, a number of practicable sufficient conditions for an operator to be totally omnipresent are furnished in Section 4, as well as a large family of examples including differential, antidifferential, integral, composition and multiplication operators.

Now, we pass to fix some notations and definitions. Throughout this paper G will stand for a domain in the complex plane \mathbb{C} and ∂G will denote its boundary taken in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. \mathbb{N} is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{Z} is the set of integers, \mathbb{R} is the real line, and $B(a, r) = \{z : |z - a| < r\}$ is the euclidean open ball with centre a and radius r ($a \in \mathbb{C}$, $r > 0$). The corresponding closed ball is $\bar{B}(a, r)$. An *operator* always refers to a continuous (not necessarily linear) self-mapping. We denote by $O(\partial G)$ the set of all open subsets of \mathbb{C}_∞ meeting ∂G . If $A \subset \mathbb{C}$ then \bar{A} (A^0) represents the closure (the interior, respectively) of A , $\|f\|_A := \sup_{z \in A} |f(z)|$, where f is a complex function defined in A , and $LT(A)$ is the set of all affine linear transformations τ , $\tau(z) = az + b$, such that $\tau(\mathbb{D}) \subset A$, where $\mathbb{D} := B(0, 1)$. As for the definition of T -monsters and of its associated notion of strongly omnipresent operators, we fix here one which is slightly stronger than that of [6], because there (as in [22]) the domain G was never \mathbb{C} in order that the finite boundary be non-empty. Nevertheless, as pointed out in [9], using chordal distances, all proofs can be adapted to the case where the boundary point under consideration is the point of infinity. Thus, as in [9], we establish the following definition.

DEFINITION 1.1. (a) A function $f \in H(G)$ is a *holomorphic monster* whenever the following universality property is satisfied: For each $g \in H(\mathbb{D})$ and each $t \in \partial G$ there exists a sequence (τ_n) of affine linear

transformations with

$$\tau_n(z) \rightarrow t \quad (n \rightarrow \infty) \text{ uniformly on } \mathbb{D} \quad \text{and} \quad \tau_n(\mathbb{D}) \subset G \quad (n \in \mathbb{N})$$

such that

$$f(\tau_n(z)) \rightarrow g(z) \quad (n \rightarrow \infty)$$

locally uniformly in \mathbb{D} .

(b) Let $T : H(G) \rightarrow H(G)$ be an operator. Then a function $f \in H(G)$ is a T -monster if Tf is a holomorphic monster. The set of T -monsters is denoted by $\mathcal{M}(T)$.

(c) An operator $T : H(G) \rightarrow H(G)$ is *strongly omnipresent* if for all $g \in H(\mathbb{D}), \varepsilon > 0, r \in (0, 1)$ and $V \in O(\partial G)$ the set

$$U(T, g, \varepsilon, r, V) := \{f \in H(G) : \text{there exists some } \tau \in LT(V \cap G) \\ \text{such that } \|(Tf) \circ \tau - g\|_{r\mathbb{D}} < \varepsilon\}$$

is dense in $H(G)$.

As in [6, Theorem 2.2], it is easy to prove that T is strongly omnipresent if and only if the set $\mathcal{M}(T)$ is residual, i.e., its complement in $H(G)$ is of first category (see also [1] for the weaker concept of omnipresent operators and [9, Example 3.4] for a linear example of an omnipresent operator which is not strongly omnipresent, such weaker operators are related to cluster sets, see [13] and [26]). Observe that an easy continuity argument allows us to restrict ourselves to *non-constant* affine linear transformations in parts (a) and (c) of the last definition. Note also that due to the results of [18, Kapitel 3] a function $f \in H(G)$ —where G is simply connected—is a holomorphic monster in the sense of Luh [22] (for future references, we call such an f a *Luh-monster*) if and only if f is simultaneously a D^j -monster and a D_a^{-j} -monster for all $j \in \mathbb{N}_0$. Here D is the differentiation operator $Df = f'$, $D^0 = I$ is the identity operator, $D^{j+1} = D \circ D^j$, a is a fixed point in the simply connected domain G , $D_a^0 = I$ and, for each $j \in \mathbb{N}$, D_a^{-j} denotes the unique antiderivative F of f of order j such that $F^{(k)}(a) = 0$ ($k \in \{0, 1, \dots, j - 1\}$). Since the intersection of countably many residual sets is again residual, the existence of Luh-monsters is thus a direct consequence of the strong omnipresence of operators D^j and $D_a^{-j}, j \in \mathbb{N}_0$. In fact, more general differential and antidifferential operators are strongly omnipresent, see [6, Sects. 3–4; 8] and Section 4. In [9] sufficient conditions are given for an operator to be strongly omnipresent, as well as characterizations of the strong omnipresence of composition and multiplication operators.

Finally, we will need in Section 3 some terminology taken from the modern theory of universality. The reader is referred to [19] for an excellent

survey about the history, results and references on this topic. If X and Y are (Hausdorff) topological vector spaces over the same field \mathbb{K} ($= \mathbb{R}$ or \mathbb{C}) and $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) is a sequence of continuous linear mappings, then (T_n) is said to be *hypercyclic* (or *universal*) whenever there is a vector $x \in X$, called also *hypercyclic* for (T_n) , such that the orbit $\{T_n x : n \in \mathbb{N}\}$ is dense in Y . Note that this forces Y to be separable. The sequence (T_n) is called *densely hypercyclic* whenever the set $HC((T_n))$ of hypercyclic vectors for (T_n) is dense. On the other hand, (T_n) is said to be *hereditarily hypercyclic* whenever (T_{n_k}) is hypercyclic for each sequence $n_1 < n_2 < n_3 < \dots$ of positive integers. The sequence (T_n) is *densely hereditarily hypercyclic* if and only if (T_{n_k}) is densely hypercyclic for every sequence $n_1 < n_2 < n_3 < \dots$ as above. For the sake of convenience, we will keep all these definitions even in the case that the mappings T_n are not linear. Finally, if $M \subset X$ is a linear manifold then we say that it is hypercyclic for (T_n) whenever $M \setminus \{0\} \subset HC((T_n))$. In [4, Theorem 2] the following result is obtained.

THEOREM 1.1. *Let X and Y be two metrizable topological vector spaces such that X is separable. Assume that $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) is a densely hereditarily hypercyclic sequence of continuous linear mappings. Then there is a dense linear submanifold of X all of whose non-zero vectors are hypercyclic for (T_n) .*

Applications of the latter theorem can be found in [4, Theorems 3–4; 20]. In fact, Theorem 1.1 is an extension of the known result of Herrero–Bourdon–Bès asserting the existence of T -invariant dense hypercyclic linear manifolds for a hypercyclic linear operator T (i.e., the sequence of iterates (T^n) is hypercyclic) on a (real or complex) locally convex space, see [10, 11, 21] (see also [5] to add the property “with maximal cardinality” to such manifolds when T acts on a Banach space).

2. TOTALLY OMNIPRESENT OPERATORS

In this section we first define in a practical way a new kind of operator. We then show how that definition can be translated in terms of approximation of vectors in certain function spaces.

Let us denote by $N(\partial G)$ the family of all sequences of similarities of the plane which take the unit disk near the boundary of G , that is,

$$N(\partial G) = \{\sigma = (\tau_n) \subset LT(G) : \tau_n \text{ is non-constant } (n \in \mathbb{N}) \text{ and} \\ \sup_{z \in \mathbb{D}} \chi(\tau_n(z), \partial G) \rightarrow 0 \ (n \rightarrow \infty)\},$$

where χ denotes the chordal distance on \mathbb{C}_∞ . Observe that since ∂G is compact in \mathbb{C}_∞ the fact $(\tau_n) \in N(\partial G)$ implies the existence of at least one boundary point t and of a sequence $\{n_1 < n_2 < n_3 < \dots\} \subset \mathbb{N}$ with $\tau_{n_k} \rightarrow t$ ($k \rightarrow \infty$) uniformly on \mathbb{D} . If T is an operator on $H(G)$, $g \in H(\mathbb{D})$, $\varepsilon > 0$, $r \in (0, 1)$ and $\sigma = (\tau_n) \in N(\partial G)$ then we set

$$U_\star(T, g, \varepsilon, r, \sigma) = \{f \in H(G) : \text{there is } n \in \mathbb{N} \text{ with } \|(Tf) \circ \tau_n - g\|_{r\mathbb{D}} < \varepsilon\}. \tag{1}$$

DEFINITION 2.1. Let $T : H(G) \rightarrow H(G)$ be an operator. We say that T is *totally omnipresent* whenever each set $U_\star(T, g, \varepsilon, r, \sigma)$ is dense in $H(G)$ ($g \in H(\mathbb{D})$, $\varepsilon > 0$, $r \in (0, 1)$, $\sigma \in N(\partial G)$).

Note that each $U_\star(T, g, \varepsilon, r, \sigma)$ is an open set of $H(G)$. For future references, we denote by $D(h, K, \delta)$ ($h \in H(G)$, $\delta > 0$, K a compact subset of G) the basic neighbourhood

$$D(h, K, \delta) = \{f \in H(G) : |f - h|_K < \delta\}. \tag{2}$$

Remark 2.1. If (g_i) is a dense sequence in $H(\mathbb{D})$ (for instance, (g_i) may be an enumeration of polynomials with coefficients having rational real and imaginary parts) then T is totally omnipresent if and only if for each $\sigma \in N(\partial G)$ and each $(i, j) \in \mathbb{N}^2$ the set $U_\star(T, g_i, \frac{1}{j}, \frac{j}{j+1}, \sigma)$ is dense in $H(G)$.

As promised, we reformulate the last definition in other language. Before this, a few more notations: If $t \in \partial G$ then $N(t)$ will stand for the set of all sequences (τ_n) of non-constant affine linear mappings with $\tau_n(\mathbb{D}) \subset G$ ($n \in \mathbb{N}$) and $\tau_n(z) \rightarrow t$ ($n \rightarrow \infty$) uniformly on \mathbb{D} . Trivially, $N(t) \subset N(\partial G)$. On the other hand, C_τ denotes composition with the function τ (i.e., $C_\tau(h) = h \circ \tau$) whenever it makes sense. In the next proposition, the equivalence (b) \Leftrightarrow (c) is trivial, but we want to establish (c) explicitly because the implication (a) \Rightarrow (c) will be crucial in the proof of Theorem 5.1.

PROPOSITION 2.2. *Let T be an operator on $H(G)$. Then the following conditions are equivalent:*

- (a) *The operator T is totally omnipresent.*
- (b) *For every $t \in \partial G$ and every $(\tau_n) \in N(t)$ there exists a dense set of functions $f \in H(G)$ satisfying that for every $g \in H(\mathbb{D})$ there exists a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $(Tf)(\tau_{n_k}(z)) \rightarrow g(z)$ ($k \rightarrow \infty$) uniformly on compact subsets of \mathbb{D} . In other words, the sequence $C_{\tau_n} \circ T : H(G) \rightarrow H(\mathbb{D})$ ($n \in \mathbb{N}$) is densely hypercyclic.*
- (c) *For every $t \in \partial G$ and every $(\tau_n) \in N(t)$, the sequence $C_{\tau_n} \circ T : H(G) \rightarrow H(\mathbb{D})$ ($n \in \mathbb{N}$) is densely hereditarily hypercyclic.*

Proof. Let (g_i) be a countable dense set in $H(\mathbb{D})$. Given $t \in \partial G$ and $\sigma = (\tau_n) \in N(t)$, the set

$$M(\sigma) := \bigcap_{i,j \in \mathbb{N}} U_{\star} \left(T, g_i, \frac{1}{j}, \frac{j}{j+1}, \sigma \right)$$

is the set of hypercyclic vectors for $\{C_{\tau_n} \circ T\}_{n \geq 1}$. So (a) \Leftrightarrow (b) follows from the fact that $H(G)$ is a Baire space. ■

We now consider the relationship between *total* and *strong* omnipresence. If we consider a set $U(T, g, \varepsilon, r, V)$ as in Definition 1.1(c) then we can associate to V a point $t \in V \cap (\partial G)$ as well as a sequence of open balls $B_n \subset G \cap V$ ($n \in \mathbb{N}$) such that $\sup_{w \in B_n} \chi(w, t) \rightarrow 0$ ($n \rightarrow \infty$). Then

$$\sup_{z \in \mathbb{D}} \chi(\tau_n(z), \partial G) \leq \sup_{z \in \mathbb{D}} \chi(\tau_n(z), t) = \sup_{w \in B_n} \chi(w, t) \rightarrow 0 \quad (n \rightarrow \infty),$$

where $\tau_n(z)$ is a non-constant affine linear mapping with $\tau_n(\mathbb{D}) = B_n$. Therefore $\sigma := (\tau_n) \in N(\partial G)$. If T is totally omnipresent then $U_{\star}(T, g, \varepsilon, r, \sigma)$ is dense in $H(G)$. If $f \in U_{\star}(T, g, \varepsilon, r, \sigma)$ then there exists $N \in \mathbb{N}$ such that $\|(Tf) \circ \tau_N - g\|_{r\mathbb{D}} < \varepsilon$, so $f \in U(T, g, \varepsilon, r, V)$ because $\tau_N \in LT(V)$ since $\tau_N(\mathbb{D}) = B_N \subset G \cap V$. Summarizing, $U_{\star}(T, g, \varepsilon, r, \sigma) \subset U(T, g, \varepsilon, r, V)$. Thus, the last set is dense. Hence we have proved that every totally omnipresent operator is strongly omnipresent.

In Section 4, we will see several examples of (linear) strongly omnipresent operators (in fact, composition operators) which are not totally omnipresent. Further examples will be provided at the end of Section 5 and after Theorem 6.1.

3. COMMON HYPERCYCLIC LINEAR MANIFOLDS

In this section we are going to improve Theorem 1.1 in order to use that improvement in Section 5. Observe that the next result asserts the existence of *common* large hypercyclic manifolds for a countable family of sequences of linear mappings. It should be pointed out that the unique additional hypothesis with respect to Theorem 1.1 is that X is Baire, which takes place, for instance, if X is complete.

THEOREM 3.1. *Let X and Y be two metrizable topological vector spaces such that X is Baire and separable. Assume that, for each $k \in \mathbb{N}$, $T_n^{(k)} : X \rightarrow Y$ ($n \in \mathbb{N}$) is a densely hereditarily hypercyclic sequence of continuous linear*

mappings. Then there is a dense linear submanifold $M \subset X$ such that

$$M \setminus \{0\} \subset \bigcap_{k \in \mathbb{N}} HC((T_n^{(k)})).$$

Proof. Observe first that hypercyclicity forces Y to be separable, so second-countable. Let us choose a dense sequence (z_n) in X and denote by d a distance on X compatible with its topology. We will consider later the open balls

$$G_N = \left\{ x \in X : d(x, z_N) < \frac{1}{N} \right\} \quad (N \in \mathbb{N}).$$

Since X is a Baire space and Y is second-countable each of the sets $HC((T_n^{(k)}))$ ($k \in \mathbb{N}$) is residual in X [19, Theorem 1], because they are dense. Therefore their intersection $\bigcap_{k \in \mathbb{N}} HC((T_n^{(k)}))$ is also residual, so dense, whence we can pick a vector

$$x_1 \in G_1 \cap \bigcap_{k \in \mathbb{N}} HC((T_n^{(k)})).$$

Then for every $k \in \mathbb{N}$ we can find a (strictly increasing) subsequence $\{p(1, k, j) : j \in \mathbb{N}\}$ of positive integers such that

$$T_{p(1,k,j)}^{(k)} x_1 \rightarrow 0 \quad (j \rightarrow \infty).$$

But, since each $(T_n^{(k)})$ ($k \in \mathbb{N}$) is densely hereditarily hypercyclic, every set $HC((T_{p(1,k,j)}^{(k)}))$ is again residual. Thus, as above, a vector x_2 can be selected in $G_2 \cap \bigcap_{k \in \mathbb{N}} HC((T_{p(1,k,j)}^{(k)}))$. Now choose for every k a subsequence $\{p(2, k, j) : j \in \mathbb{N}\}$ of $\{p(1, k, j)\}$ with

$$T_{p(2,k,j)}^{(k)} x_2 \rightarrow 0 \quad (j \rightarrow \infty).$$

Note that also $T_{p(2,k,j)}^{(k)} x_1 \rightarrow 0$ ($j \rightarrow \infty$) for each $k \in \mathbb{N}$. Since the new sequences $(T_{p(2,k,j)}^{(k)})$ ($k \in \mathbb{N}$) are again densely hypercyclic, one can choose a vector $x_3 \in G_3 \cap \bigcap_{k \in \mathbb{N}} HC((T_{p(2,k,j)}^{(k)}))$.

It is evident that this process can be continued by induction, getting a sequence $\{x_N : N \in \mathbb{N}\} \subset X$ and a family $\{\{p(n, k, j) : j \in \mathbb{N}\} : n, k \in \mathbb{N}\}$ of sequences of positive integers satisfying

$$x_N \in G_N \quad \text{for all } N \in \mathbb{N}, \quad (3)$$

$$x_N \in \bigcap_{k \in \mathbb{N}} HC((T_{p(N-1,k,j)}^{(k)})) \quad \text{for all } N \in \mathbb{N} \quad (4)$$

and

$$T_{p(n,k,j)}^{(k)} x_N \rightarrow 0 \quad (j \rightarrow \infty) \quad \text{for all } n \geq N \text{ and all } k \in \mathbb{N}, \quad (5)$$

where, in order to make the notation consistent, $(p(0, k, j))$ stands for the whole sequence of positive integers for every $k \in \mathbb{N}$. Define

$$M = span(\{x_N : N \in \mathbb{N}\}).$$

Since $\{z_n : n \in \mathbb{N}\}$ is dense in X and $d(x_n, z_n) < \frac{1}{n} \rightarrow 0$ ($n \rightarrow \infty$) (by (3)), the set $\{x_n : n \in \mathbb{N}\}$ is also dense, hence M is a dense linear submanifold of X .

It remains to prove that each non-zero vector of M is hypercyclic for each sequence $(T_n^{(k)})$ ($k \in \mathbb{N}$). Fix $x \in M \setminus \{0\}$. Then there are finitely many scalars a_1, \dots, a_N with $a_N \neq 0$ such that $x = \sum_{n=1}^N a_n x_n$. Since a non-zero multiple of a hypercyclic vector is still hypercyclic, we may assume that $a_N = 1$. Fix a positive integer k and a vector $y \in Y$. Let us show a subsequence $\{T_{r(j)}^{(k)} : j \in \mathbb{N}\}$ of $(T_n^{(k)})$ such that

$$T_{r(j)}^{(k)} x \rightarrow y \quad (j \rightarrow \infty).$$

By (4), there is a subsequence $(r(j))$ of $(p(N - 1, k, j))$ such that

$$T_{r(j)}^{(k)} x_N \rightarrow y \quad (j \rightarrow \infty).$$

But, since $(r(j))$ is a subsequence of $(p(N - 1, k, j))$ we see from (5) that $T_{r(j)}^{(k)} x_n \rightarrow 0$ ($j \rightarrow \infty$) for all $n \in \{1, \dots, N - 1\}$, so $\sum_{n=1}^{N-1} a_n T_{r(j)}^{(k)} x_n \rightarrow 0$ ($j \rightarrow \infty$). Finally, by (6) and linearity,

$$T_{r(j)}^{(k)} x = T_{r(j)}^{(k)} x_N + \sum_{n=1}^{N-1} a_n T_{r(j)}^{(k)} x_n \rightarrow y + 0 = y \quad (j \rightarrow \infty),$$

as required. ■

4. SUFFICIENT CRITERIA FOR TOTAL OMNIPRESENCE AND EXAMPLES

The organization of this section is as follows. First, we establish the total omnipresence of differential, antidifferential and integral operators under rather general conditions, see Theorem 4.1. This supplies a large class of

examples, including the operators D^N and D_a^{-N} ($N \in \mathbb{N}_0$). In particular, *the identity operator I becomes totally omnipresent*. Second, we will construct new totally omnipresent operators from known ones. As an application we will see that every onto linear operator on $H(G)$ is totally omnipresent. Next we study the following problem: Under which conditions does the existence of a single T -wild-behaved function associated to each boundary point and each sequence of affine transformations coming near that point suffice to make T totally omnipresent? Afterwards, we provide with some workable conditions under which a general operator is totally omnipresent. Finally, we apply some of these results to furnish new examples of this kind of operators. In fact, we will be able to characterize the total omnipresence for left-composition operators and multiplication operators. For right-composition operators, necessary conditions and sufficient conditions are given. It happens that in many cases the criteria as well as the examples to be given for total omnipresence are close to those of strong omnipresence. Hence we will simplify (or even drop, as in Theorem 4.1) the proof of each result about total omnipresence whenever it is very similar to that of the corresponding strong omnipresence result, see [6, 8, 9].

Let us start with the definition of the operators to be handled. Let $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function of subexponential type, that is, for every $\varepsilon > 0$ there is a constant $M = M(\varepsilon) > 0$ such that $|\Phi(z)| \leq M e^{\varepsilon|z|}$ for all $z \in \mathbb{C}$. Then the associated linear differential operator $\Phi(D) = \sum_{j=0}^{\infty} a_j D^j$ is well defined on $H(G)$. This still holds if Φ is just of exponential type (i.e., there are constants $M, K > 0$ such that $|\Phi(z)| \leq M e^{K|z|}$ for all $z \in \mathbb{C}$) whenever $G = \mathbb{C}$. If G is a simply connected domain, $a \in G$ and $\varphi : G \times G \rightarrow \mathbb{C}$ is a function that is holomorphic in both variables, then the Volterra integral operator of the first kind $V_\varphi f(z) = \int_a^z f(t)\varphi(z, t) dt$ ($z \in G$) makes sense on $H(G)$. In particular, by choosing $\varphi(z, t) := \sum_{j=1}^{\infty} a_j \frac{(z-t)^{j-1}}{(j-1)!}$, we obtain that if $\Psi(z) = \sum_{j=1}^{\infty} a_j z^j$ is a formal complex power series such that

$$\limsup_{j \rightarrow \infty} \frac{|a_j|^{1/j}}{j} \leq \frac{1}{\Delta(a, G)},$$

where $\Delta(a, G) = \sup_{z \in G} \inf\{r > 0 : a \text{ is in the connected component of } B(z, r) \cap G \text{ containing } z\}$, then the associated linear antidifferential operator $\Psi(D_a^{-1}) = \sum_{j=1}^{\infty} a_j D_a^{-j}$ is well defined on $H(G)$, see [3]. On the other hand, we recall that if $\varphi \in H(G, G) := \{f \in H(G) : f(G) \subset G\}$, ψ is an entire function and $h \in H(G)$ then the, respectively, associated right-composition operator C_φ , left-composition operator L_ψ and multiplication operator M_h are defined on $H(G)$ as

$$C_\varphi(f) = f \circ \varphi, \quad L_\psi(f) = \psi \circ f, \quad M_h(f) = h \cdot f.$$

As for differential and integral operators, we have that under weak hypotheses all of them and some combinations of them are strongly omnipresent (see [6, Sects. 3–4; 8, Sects. 2–3]). But observe that in many proofs a set $U(T, g, \varepsilon, r, V)$ as in Definition 1.1 as well as a neighbourhood as in (2) are fixed. Then a suitable $\tau \in LT(V \cap G)$ is found in order to get the density of that set. A simple glance reveals that we can in fact fix a point $t \in \partial G$ and a sequence $\sigma = (\tau_n) \in N(t)$ in such a way that a positive integer n is available (with $\tau_n(\mathbb{D})$ close to t enough) to make $U_\star(T, g, \varepsilon, r, \sigma)$ dense. Consequently, we can establish the following theorem.

THEOREM 4.1. *Assume that $\Phi(z), \Psi(z)$ are power series as above and that $\varphi : G \times G \rightarrow \mathbb{C}$ is a holomorphic function in both variables. Let a be a fixed point in G . Suppose also that P is a polynomial and that, if $N \in \mathbb{N}_0$, $c_j(z)$ ($j = 0, \dots, N$) are holomorphic functions on G . We have:*

- (a) *If Φ is non-zero then the operator $\Phi(D)$ is totally omnipresent.*
- (b) *If G is simply connected and Ψ is non-zero then $\Psi(D_a^{-1})$ is totally omnipresent.*
- (c) *If G is simply connected and either Φ or P is non-zero then the operator $\Phi(D) + P(D_a^{-1})$ is totally omnipresent.*
- (d) *If G is simply connected and $c_N(z) \neq 0$ for all $z \in G$ except for a finite subset of G then the operator T on $H(G)$ defined by*

$$Tf(z) = \sum_{j=0}^N c_j(z)f^{(j)}(z) + V_\varphi(z) \quad (f \in H(G), z \in G)$$

is totally omnipresent. In particular, if P is non-zero then the operators $P(D) + V_\varphi$ and $P(D) + \Psi(D_a^{-1})$ are totally omnipresent.

Specifically, part (a) ((b), (c), (d), respectively) follows after modifying suitably the proof of [6, Theorem 3.1] ([6, Theorem 4.2, Corollary 4.3; 8, Theorems 3.6, 3.4], respectively).

As proposed in [8], the strong omnipresence (so the total omnipresence) of $\Phi(D) + \Psi(D_a^{-1})$ is unknown to us up to date. As for part (d) of the last theorem, observe that even for φ non-zero the operator V_φ may not be totally omnipresent (see Section 6), and that T may not be totally omnipresent if c_N is just supposed to be non-zero (for an example, see the last paragraph of Section 4, where we take $N = 0, \varphi = 0$). Nevertheless, they are strongly omnipresent, see [8]. The point is that if we try to adapt the proof of [8] to the total omnipresence of these V_φ and T then one sees that one cannot start with a prefixed sequence (τ_n) with $(\tau_n(\mathbb{D}))$ close to the boundary. We will go back to these operators in Section 6.

Next we state a remark containing the promised assertion on onto linear operators. We denote $TS := T \circ S$. On the other hand, the product $T \cdot S$ is defined as $(T \cdot S)f = Tf \cdot Sf$.

Remark 4.2. Let T and S be operators on $H(G)$, with T totally omnipresent.

(i) Suppose that each pre-image $S^{-1}(\Omega)$ is dense in $H(G)$ whenever Ω is. Then TS and S are totally omnipresent. In particular, by the Open Mapping Theorem, this occurs when S is both linear and onto.

(ii) Assume that for every $t \in \partial G$ and every $f \in H(G)$ there exists

$$\lim_{z \rightarrow t} (Sf)(z) \in \mathbb{C} \quad (\text{respectively, } \mathbb{C} \setminus \{0\}).$$

Then $T + S$ (respectively, $T \cdot S$) is totally omnipresent.

Proof. Part (ii) is easy and left as an exercise to the reader. As for (i), let $t \in \partial G$ and $\sigma = (\tau_n) \in N(t)$. By hypothesis,

$$HC((C_{\tau_n} TS)) = S^{-1}(HC((C_{\tau_n} T)))$$

is dense in $H(G)$. So TS is totally omnipresent, by Proposition 2.2(b). To see now that S is totally omnipresent, apply the case when $T = I$. ■

For example, if G is simply connected and a is a fixed point in G , then the differentiation operator D (and so D^N) is (linear and) onto on $H(G)$. Therefore we derive that the operator $R_{N,a}$ on $H(G)$ given by

$$R_{N,a}f(z) = f(z) - \sum_{j=0}^{N-1} \frac{f^{(j)}(a)}{j!} (z - a)^j,$$

that is, the value at z of Taylor's remainder of order N of f at a , is totally omnipresent: just take $T = D_a^{-N}$ and $S = D^N$ in part (i) of Remark 4.2. Note that neither D_a^{-N} nor $R_{N,a}$ is onto; they do not even have dense range.

Consider again the operator $R_{N,a}$ acting on $H(G)$, where G is any bounded domain (G may be non-simply connected this time). Then we can write $R_{N,a} = T + S$ with $T = I$ and $Sf(z) = - \sum_{j=0}^{N-1} \frac{f^{(j)}(a)}{j!} (z - a)^j$, whence part (ii) of Remark 4.2 applies, yielding $R_{N,a}$ again as a total omnipresent operator in this new situation.

We now state a sufficient condition under which the existence for each boundary point of a single wild function with respect to a linear operator yields total omnipresence, compare [9, Theorem 2.7]. At this point it is convenient to introduce the following definition. We say that an operator T on $H(G)$ is ∂ -hypercyclic if and only if for each $t \in \partial G$ and each

$\sigma = (\tau_n) \in N(t)$ the sequence $(C_{\tau_n}T)$ is hypercyclic. It is evident from Proposition 2.2 that total omnipresence implies ∂ -hypercyclicity.

PROPOSITION 4.3. *Let T be a linear ∂ -hypercyclic operator on $H(G)$ such that for each $t \in \partial G$ there is a dense subset $\mathcal{D}_t \subset H(G)$ satisfying that there exists*

$$\lim_{z \rightarrow t} (Th)(z) \in \mathbb{C}$$

for every $h \in \mathcal{D}_t$. Then T is totally omnipresent.

Proof. For each boundary point t and each sequence $(\tau_n) \in N(t)$ we fix a hypercyclic function f_t for $(C_{\tau_n}T)$. Then the set $f_t + \mathcal{D}_t$ is dense and it is contained in $HC((C_{\tau_n}T))$. Indeed, \mathcal{D}_t is dense and for fixed $h \in \mathcal{D}_t$ and $g \in H(\mathbb{D})$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ for which $Tf_t(\tau_{n_k}(z)) \rightarrow g(z) - \alpha(t)$ ($k \rightarrow \infty$) in $H(\mathbb{D})$, where $\alpha(t)$ is the limit guaranteed by the hypothesis; therefore $(T(f_t + h)) \circ \tau_{n_k}$ tends to g . Then Proposition 2.2 applies and we are done. ■

For instance, the condition in the above theorem is satisfied by a differential operator $\Phi(D)$ and by a finite order antidifferential operator $P(D_a^{-1})$ (P is a polynomial and $a \in G$) whenever G is a bounded simply connected domain. Indeed, choose $\mathcal{D}_t = \{\text{polynomials}\}$ for all $t \in \partial G$.

If linearity is *not* imposed on T , different additional hypotheses about the behaviour of T near the boundary are needed. We will say that an operator T on $H(G)$ is *locally stable near the boundary* whenever the following property is satisfied: For each compact subset K of G there exists a compact subset M of G such that for each closed ball $B \subset G \setminus M$, each $f \in H(G)$ and each $\varepsilon > 0$ there exist a compact set $S \subset G \setminus K$ with $\mathbb{C} \setminus S$ connected and $\delta > 0$ such that if $g \in H(G)$ and $\|f - g\|_S < \delta$ then $\|Tf - Tg\|_B < \varepsilon$.

It should be pointed out that our definition of local stability is less restrictive than that given in [9, Definition 2.1]. There the set S was a closed ball, but a close look at the proofs reveals that the connectivity of $\mathbb{C} \setminus S$ is all that is needed.

For instance, from Cauchy’s integral formula for derivatives, it is easy to verify that each differential operator $\Phi(D)$ is locally stable near the boundary; in fact we can always take concentric balls B, S with $\text{radius}(B) < \text{radius}(S)$.

Due to the same considerations given just before our Theorem 4.1, we establish without proof the following result, cf. [9, Theorem 2.6].

THEOREM 4.4. *Let T be a ∂ -hypercyclic operator on $H(G)$ that is locally stable near the boundary. Then T is totally omnipresent.*

It happens that, at least for the non-linear case (if T is linear the answer is unknown to us), $\mathcal{M}(T)$ may be non-empty while T is not strongly omnipresent, see [9, Example 2.8]. Unfortunately, we do not know whether the corresponding result holds in the new setting of this paper. Accordingly, we raise the following question:

Is every ∂ -hypercyclic operator totally omnipresent?

Our next goal is to get practicable conditions on an operator that guarantee its total omnipresence. Combining the first hypothesis of Theorem 4.4 with the following notion will give positive results. Following [9], we say that T has *locally dense range near the boundary* if there exists a compact subset M of G such that for each open ball $U \subset G \setminus M$, the restriction operator $T_U : f \in H(G) \mapsto (Tf)|_U \in H(U)$ has dense range. Every operator with dense range has, trivially, locally dense range. For instance, every non-zero differential operator $\Phi(D)$ has dense range whenever G is simply connected since $\Phi(D)$ is onto on the space of entire functions $H(\mathbb{C})$ [14, 25] and $H(\mathbb{C})$ is dense in $H(G)$. The same reasoning shows that $\Phi(D)$ has *locally dense range in any domain* G . Also the antidifferential operator D_a^{-N} has locally dense range near the boundary. As the linear example $Tf(z) = f(z/2)$ (with $G = \mathbb{D}$) shows (see [9, Example 2.10]), the density of the range does not imply strong (so total) omnipresence. On the other hand, D_a^{-N} tells us that an operator with non-dense range may be totally omnipresent. The trick is in the fact that it possesses both “local” properties, namely, density and stability.

THEOREM 4.5. *Let T be an operator on $H(G)$ having locally dense range and local stability near the boundary. Then T is totally omnipresent.*

Proof. Let us fix sets $U_\star(T, g, \varepsilon, r, \sigma)$ and $D(h, K, \delta)$ as in Definition 2.1 and (2), respectively. Our goal is to show that their intersection is not empty. Put $\sigma = (\tau_n)$ and retain in mind that $\sigma \in N(\partial G)$. We have that $\tau_n(\mathbb{D}) = B(z_n, r_n)$ for certain $z_n \in \mathbb{C}$, $r_n > 0$ ($n \in \mathbb{N}$). Denote by B_n the closure of these balls ($n \in \mathbb{N}$). By hypothesis, there exists a compact subset $M \subset G$ such that for each open ball $U \subset G \setminus M$, the mapping $f \in H(G) \mapsto (Tf)|_U \in H(U)$ has dense range. Without loss of generality, it can be supposed that the compact set M is the same as that given for K in the definition of local stability. Since the balls B_n approach the boundary, one can select a positive integer N with $B_N \subset G \setminus M$. Thus, if one chooses $U = B(z_N, r_N)$ then there exists a function $f \in H(G)$ such that

$$\|Tf - g \circ \tau_N^{-1}\|_{\tau_N(r_N\mathbb{D})} < \frac{\varepsilon}{2}, \tag{7}$$

because $g \circ \tau_N^{-1} \in H(U)$. Now the local stability comes in our help, yielding a compact set $S \subset G \setminus K$ with $\mathbb{C} \setminus S$ connected and a $\delta_1 > 0$ such that for all $\varphi \in H(G)$

$$\|\varphi - f\|_S < \delta_1 \quad \text{implies that} \quad \|T\varphi - Tf\|_{B_N} < \frac{\varepsilon}{2}. \tag{8}$$

By Runge’s theorem, it is possible to find a function $f_1 \in H(G)$ (in fact, a rational one with poles outside G , as the method of the pole-pushing shows, see for instance [15]) satisfying

$$\|f_1 - h\|_K < \delta \tag{9}$$

and

$$\|f_1 - f\|_S < \delta_1.$$

To achieve this, we have taken K with the property that each connected component of $\mathbb{C}_\infty \setminus K$ contains at least one connected component of $\mathbb{C}_\infty \setminus G$ (so $K \cup S$ enjoys the same property), which carries no loss of generality. By (8),

$$\|Tf_1 - Tf\|_{B_N} < \frac{\varepsilon}{2}. \tag{10}$$

Now, (7), (10) and the fact that $\tau_N(r\bar{\mathbb{D}}) \subset B_N$ yield

$$\|Tf_1 - g \circ \tau_N^{-1}\|_{\tau_N(r\bar{\mathbb{D}})} \leq \|Tf_1 - Tf\|_{B_N} + \|Tf - g \circ \tau_N^{-1}\|_{\tau_N(r\bar{\mathbb{D}})} < \varepsilon. \tag{11}$$

Consequently, from (9) and (11),

$$f_1 \in U_\star(T, g, \varepsilon, r, \sigma) \cap D(h, K, \delta),$$

and we are done. ■

Remark 4.6. A closer look at the last proof (a suitable subsequence of (τ_n) tending to some boundary point will be needed) reveals that in order that T be totally omnipresent it suffices that the following property holds: For each compact subset $K \subset G$ and each $t \in \partial G$ there is an open set V with $V \ni t$ such that, for every closed ball $B \subset V \cap G$,

(i) The restriction mapping T_{B^0} has dense range and

(ii) For each $f \in H(G)$ and $\varepsilon > 0$ there exist a compact set $S \subset G \setminus K$ with connected complement and $\delta > 0$ such that for all $g \in H(G)$ the fact $\|f - g\|_S < \delta$ implies $\|Tf - Tg\|_B < \varepsilon$.

Note that if T is linear then (ii) reduces to say

(ii') For each $\varepsilon > 0$ there exist a compact set $S \subset G \setminus K$ with connected complement and $\delta > 0$ such that if $g \in H(G)$ and $\|g\|_S < \delta$ then $\|Tg\|_B < \varepsilon$.

Only a piece of caution: the weaker notions of “somewhere local stability” and “somewhere local density” introduced in [9] do *not* work here, because they do not permit to fix a sequence (τ_n) tending to a given boundary point.

As a consequence of Theorem 4.5 we obtain that, in particular, if T is an onto locally stable operator (not necessarily linear) then it is totally omnipresent, compare with Remark 4.2. In addition, we derive again, independently, that the identity operator is totally omnipresent.

We are now passing to study the total omnipresence of the right-composition operator C_φ generated by a holomorphic self-mapping $\varphi \in H(G, G)$. Its strong omnipresence has been recently characterized in [9, Theorem 3.1]. Specifically, it is proved there that C_φ is strongly omnipresent if and only if $\mathcal{M}(C_\varphi)$ is non-empty if and only if for every $V \in O(\partial G)$ the set $\varphi(V \cap G)$ is not relatively compact in G . In particular, if $G = \mathbb{C}$, then C_φ is strongly omnipresent if and only if φ is non-constant. Unfortunately, we have not been able this time to isolate the exact conditions for C_φ to be totally omnipresent, see Theorems 4.8 and 4.9. At this point it is convenient to introduce a new concept and to recall a topological notion.

DEFINITION 4.1. We say that a function $F : G \rightarrow \mathbb{C}$ is *locally one-to-one near the boundary* if and only if there is a compact set $K \subset G$ such that F is one-to-one on every open ball $U \subset G \setminus K$.

By using the compactness of ∂G in \mathbb{C}_∞ , it is easy to see that F is locally one-to-one near the boundary if and only if we can associate to each $t \in \partial G$ an open set $V \subset \mathbb{C}_\infty$ containing t satisfying that F is one-to-one on every open ball $U \subset V \cap G$. A mapping $F : X \rightarrow Y$ between two topological spaces X, Y is called *proper* if the preimage $F^{-1}(K)$ of each compact subset $K \subset Y$ is compact in X . In our setting, the following lemma will reveal itself to be useful.

LEMMA 4.7. *A continuous self-mapping $F : G \rightarrow G$ is proper if and only if for each $t \in \partial G$ and every compact set $K \subset G$ there exists an open set $V \subset \mathbb{C}_\infty$ with $V \ni t$ such that $F(V \cap G) \cap K = \emptyset$.*

Proof. Assume that F is proper and that, by the way of contradiction, there exist a boundary point t and a compact subset $K \subset G$ with the property that $F(V_n \cap G) \cap K \neq \emptyset$ for all $n \in \mathbb{N}$, where V_n is the chordal ball in \mathbb{C}_∞ with centre t and radius $1/n$. Then we can select a point $z_n \in V_n \cap G$

such that $F(z_n) \in K$. Hence $F^{-1}(K)$ is not compact, because $(z_n) \subset F^{-1}(K) \subset G$ but $z_n \rightarrow t \in \partial G$ as $n \rightarrow \infty$. This is a contradiction.

Conversely, assume F is not proper, that is, that there exists a compact subset $K \subset G$ such that $F^{-1}(K)$ is not compact. But, by continuity, $F^{-1}(K)$ is closed in G , so $F^{-1}(K)$ cannot be relatively compact in G . Hence there exist a boundary point t and a sequence $(z_n) \subset F^{-1}(K)$ with $z_n \rightarrow t$ as $n \rightarrow \infty$. Given any open set $V \subset \mathbb{C}_\infty$ containing t , we can choose $n_0 \in \mathbb{N}$ satisfying $z_n \in V \cap G$ for all $n > n_0$. Therefore $F(z_n) \in F(V \cap G) \cap K$ for all $n > n_0$, which tells us that $F(V \cap G) \cap K \neq \emptyset$. This concludes the proof. ■

In our next theorem we will show how the latter two properties—which are rather practicable—suffice for total omnipresence. In Theorem 4.9 we get at least that the fact that φ be proper is necessary, but the local bijectivity should be changed to a kind of (not very pleasant) “ $(\frac{1}{3})$ -local bijectivity near the boundary,” see Theorem 4.9.

THEOREM 4.8. *If φ is proper and locally one-to-one near the boundary then the operator C_φ is totally omnipresent.*

Proof. We will try to apply Theorem 4.5, or rather Remark 4.6 after it, with $T = C_\varphi$. Hence, our goal is to show that (i) and (ii') are fulfilled. Fix $t \in \partial G$ and a compact set $K \subset G$. By hypothesis and by Lemma 4.7, there exists an open set $V_1 \subset \mathbb{C}_\infty$ with $V_1 \ni t$ such that

$$\varphi(V_1 \cap G) \cap K = \emptyset.$$

Since φ is locally one-to-one near the boundary there is another open set V_2 with $t \in V_2 \subset V_1$ satisfying that φ is one-to-one on every open ball $U \subset V_2 \cap G$.

We show now that $V = V_2$ satisfies (i) and (ii'). Fix a closed ball $B \subset V_2 \cap G$. Then there is an open ball U with $B \subset U \subset V_2 \cap G$. If $S := \varphi(B)$, then $S \subset \varphi(V_2 \cap G) \subset \varphi(V_1 \cap G) \subset G \setminus K$, and $\mathbb{C} \setminus S$ is connected because $\varphi : U \rightarrow \varphi(U)$ is an isomorphism. It is clear that given $\varepsilon > 0$ and $g \in H(G)$ with $\|g\|_S < \delta := \varepsilon$ then

$$\|C_\varphi g\|_B = \|g \circ \varphi\|_B = \|g\|_S < \varepsilon.$$

On the other hand, the restriction mapping $(C_\varphi)_{B^0} : f \in H(G) \mapsto (f \circ \varphi)|_{B^0} \in H(B^0)$ has dense range, because $\varphi : B^0 \rightarrow \varphi(B^0)$ is an isomorphism and $H(G)$ is dense in $H(\varphi(B^0))$ by Runge's theorem (note that $\varphi(B^0)$ is a simply connected domain contained in G). Thus, (i) and (ii') are satisfied, and the proof is concluded. ■

Recall that every totally omnipresent operator is ∂ -hypercyclic by Proposition 2.2.

THEOREM 4.9. *Assume that $\varphi \in H(G, G)$ and that C_φ is ∂ -hypercyclic. Then φ is proper and satisfies the following property:*

For every real number $s > 3$ there exists a compact set $K \subset G$ such that, for every open ball $B(a, R) \subset G \setminus K$, φ is one-to-one on $B(a, R/s)$.

Proof. Suppose, by the way of contradiction, that φ is not proper. Therefore, by Lemma 4.7, there is a boundary point t and a compact set $K \subset G$ such that for each $n \in \mathbb{N}$ we can select a point $z_n \in V_n \cap G$ with $\varphi(z_n) \in K$, where V_n is the chordal ball with centre t and radius $1/n$. For every n we can choose $r_n > 0$ such that $B(z_n, r_n) \subset V_n \cap G$. Define $\sigma = (\tau_n)$ as $\tau_n(z) = r_n z + z_n$. Then $\tau_n(\mathbb{D}) = B(z_n, r_n) \subset V \cap G$ and

$$\sup_{z \in \mathbb{D}} \chi(t, \tau_n(z)) \leq \frac{1}{n} \rightarrow 0 \quad (n \rightarrow \infty),$$

therefore $\sigma \in N(t)$. Consider a function $f \in H(G)$ and the constant function $g(z) := 1 + M$, where $M = \max_{z \in K} |f(\varphi(z))|$. Then $g \in H(\mathbb{D})$ and, for all $n \in \mathbb{N}$ and all $r > 0$,

$$\begin{aligned} \|C_{\tau_n} C_\varphi f - g\|_{r\bar{\mathbb{D}}} &\geq |f(\varphi(z_n)) - 1 - M| \\ &\geq M + 1 - |f(\varphi(z_n))| \geq 1. \end{aligned}$$

Hence $(C_{\tau_n} C_\varphi)$ cannot be hypercyclic, which is a contradiction.

Assume now, again by the way of contradiction, that φ does not satisfy the $(\frac{1}{3})$ -property given in the statement. Then there are a real number $s > 3$ and a sequence of balls $B(a_n, r_n) \subset G$ tending to the boundary in such a way that φ is not one-to-one on $B(a_n, \frac{r_n}{s})$. By taking a subsequence if necessary, we can suppose that

$$\sup_{z \in B(a_n, r_n)} \chi(t, z) \rightarrow 0 \quad (n \rightarrow \infty) \tag{12}$$

for some boundary point t . For each positive integer n there exist points $z_n, w_n \in B(a_n, \frac{r_n}{s})$ satisfying $z_n \neq w_n$ and $\varphi(z_n) = \varphi(w_n)$. Consider the following sequence $\sigma = (\tau_n)$ of affine linear transformations:

$$\tau_n(z) = \frac{s}{3}(w_n - z_n)z + z_n \quad (n \in \mathbb{N}).$$

Then $\tau_n(0) = z_n$, $\tau_n(\frac{2}{3}) = w_n$ and $\tau_n(\mathbb{D}) = B(z_n, \frac{s}{3}|w_n - z_n|) \subset B(z_n, \frac{s}{3} \cdot \frac{2r_n}{s}) \subset B(a_n, r_n)$. Consequently, by (12),

$$\sup_{z \in \mathbb{D}} \chi(t, \tau_n(z)) \rightarrow 0 \quad (n \rightarrow \infty),$$

that is, $\sigma \in N(t)$. By hypothesis, there must be a function $f \in HC((C_{\tau_n} C_\varphi))$. Thus, for a suitable subsequence (τ_{n_j}) of (τ_n) , $(C_{\tau_{n_j}} C_\varphi f)$ tends to the identity function $g(z) = z$ in $H(\mathbb{D})$. In particular, $f(\varphi(\tau_{n_j}(0))) \rightarrow 0$ and $f(\varphi(\tau_{n_j}(3/s)))$

$\rightarrow 3/s$ as $j \rightarrow \infty$. But this would yield that $f(\varphi(z_{n_j})) \rightarrow 0$ and $f(\varphi(w_{n_j})) \rightarrow 3/s$ ($j \rightarrow \infty$), which is a contradiction because both sequences are the same. ■

In the case $G = \mathbb{C}$ the following corollary is derived from the latter two theorems.

COROLLARY 4.10. *Let φ be an entire function. We have:*

- (a) *If C_φ is ∂ -hypercyclic then φ is a non-constant polynomial.*
- (b) *If φ is a polynomial of degree one or two then C_φ is totally omnipresent.*

Proof. Due to Picard’s theorem [17, Chap. 9] and to the fact that $\lim_{z \rightarrow \infty} P(z) = \infty$ if P is a non-constant polynomial, only these polynomials are proper, hence the transcendental entire functions are excluded from ∂ -hypercyclicity by Theorem 4.9. This proves (a). As for (b), if φ is a polynomial of degree one, then φ is bijective from \mathbb{C} onto \mathbb{C} , so it is locally one-to-one near the boundary and Theorem 4.8 applies. Assume that φ is a polynomial of degree two, namely, $\varphi(z) = az^2 + bz + c$ ($a, b, c \in \mathbb{C}; a \neq 0$). Our goal is to get a compact set $K \subset \mathbb{C}$ such that φ is one-to-one on every open ball $U \subset \mathbb{C} \setminus K$. Choose $K = \bar{B}(0, 1 + |b/a|)$. Then a ball U as before would lie on a half-plane H which is at a distance greater than $|b/a|$ from the origin. A simple calculation shows that if $\varphi(z) = \varphi(w)$ and $z \neq w$ then $w = -\frac{b}{a} - z$. But if $z \in U$ then $z \in H$, whence $w \in -\frac{b}{a} - H$. Hence $w \notin U$ because $H \cap (-\frac{b}{a} - H) = \emptyset$. Thus, φ is one-to-one on U , as required. ■

By Theorem 4.9 and Corollary 4.10 we can furnish a new (even linear) example of a strongly omnipresent operator which it is not totally omnipresent.

EXAMPLE 4.11. Choose $G = \mathbb{D}$ and let φ be the Blaschke product with zeros at the points $z_n = 1 - \frac{1}{n^2}$ ($n \in \mathbb{N}$), that is,

$$\varphi(z) = \prod_{n=1}^{\infty} \frac{n^2 z - n^2 + 1}{(n^2 - 1)z - n^2} \quad (z \in \mathbb{D}).$$

Since $\sum(1 - |z_n|) < +\infty$, we have (see [16, Theorem 6.1]) that $\varphi \in H(\mathbb{D}, \mathbb{D})$ and that φ extends to a continuous function on $\bar{\mathbb{D}} \setminus \{1\}$ with $|\varphi(z)| = 1$ on $(\partial\mathbb{D}) \setminus \{1\}$. Then $(\partial\mathbb{D}) \cap \partial\varphi(V \cap G)$ is not empty for all $V \in O(\partial\mathbb{D})$, hence $\varphi(V \cap G)$ is not relatively compact in \mathbb{D} , so C_φ is strongly omnipresent. However, C_φ is not totally omnipresent because φ is not proper, since, for instance, $\varphi^{-1}(\{0\}) = \{z_n : n \in \mathbb{N}\}$, which is not compact. In the case $G = \mathbb{C}$ any C_φ with φ transcendental is strongly omnipresent but not totally omnipresent.

As for the case when φ is a polynomial, we raise the following.

CONJECTURE 4.12. Let φ be a polynomial of degree 3 or larger. Then C_φ is not totally omnipresent.

We shall be content for now by proving that if N is a positive integer with $N \geq 10$ and $\varphi(z) = z^N$ then C_φ is not totally omnipresent. According to Theorem 4.9, this will be achieved as soon as we can show a real number $s > 3$ in such a way that to each $r > 0$ we can associate a ball $B(a, R) \subset \{|z| > r\}$ with the property that φ is not one-to-one on $B(a, R/s)$. Since $\sin \frac{\pi}{5} < 0.31$, we have that $\sin \frac{2\pi}{N} \leq \sin \frac{\pi}{5} < \frac{1}{3}$. Choose s with $\sin \frac{\pi}{N} < \frac{1}{s} < \frac{1}{3}$, and fix $r > 0$. Since $\frac{R/s}{R+r} \rightarrow \frac{1}{s}$ as $R \rightarrow \infty$, we can select an $R > 0$ with $\sin \frac{\pi}{N} < \frac{R/s}{R+r}$. Consider the balls $B(a := R+r, R)$, and $B(R+r, R/s)$. The first one is in $\{|z| > r\}$ while the second one is tangent to two rays from the origin making an angle of opening $2 \arcsin \frac{R/s}{R+r}$, which is greater than $2\pi/N$. But given $w_0 \in \mathbb{C} \setminus \{0\}$ the roots of $\varphi(z) = w_0$ are N points in the circle $|z| = |w_0|^{1/N}$ equally distributed with angular distance equal to $2\pi/N$. Then for a suitable w_0 there are at least two roots of $\varphi(z) = w_0$ in the second ball, that is, φ is not one-to-one on $B(a, R/s)$.

We are now assuming that L_ψ is the left-composition operator on $H(G)$ associated to an entire function ψ . In [9, Sect. 3] it is asserted that L_ψ is strongly omnipresent on $H(G)$ if and only if $\mathcal{M}(L_\psi) \neq \emptyset$ if and only if ψ has an *approximate right inverse*, that is, there is a sequence (f_n) of entire functions such that $\psi(f_n(z)) \rightarrow z$ ($n \rightarrow \infty$) locally uniformly in \mathbb{C} . The proof is based there on the fact that locally dense range plus local stability near the boundary imply strong omnipresence. But they also imply total omnipresence, see Theorem 4.5. Consequently, we are allowed to establish the following theorem.

THEOREM 4.13. *Let L_ψ be the left-composition operator on $H(G)$ defined by $\psi \in H(\mathbb{C})$. Then the following assertions are equivalent:*

- (a) *The operator L_ψ is totally omnipresent.*
- (b) *The operator L_ψ is strongly omnipresent.*
- (c) *The operator L_ψ is ∂ -hypercyclic.*
- (d) *The set $\mathcal{M}(L_\psi)$ is non-empty.*
- (e) *The function ψ has an approximate right inverse.*

In particular, ψ must be surjective in order that L_ψ be totally omnipresent; but surjectivity alone is not sufficient (see [9]).

We finish this section by considering the multiplication operator M_h generated by a function $h \in H(G)$. In [9] it is shown that M_h is strongly omnipresent if and only if h is non-zero. It is easy to realize that a stronger condition is needed for total omnipresence.

THEOREM 4.14. *Assume that $h \in H(G)$. Then the following are equivalent:*

- (a) *The operator M_h is totally omnipresent.*
- (b) *The operator M_h is ∂ -hypercyclic.*
- (c) *The set of zeros of h is finite.*

Proof. That (a) implies (b) is due to Proposition 2.2. Suppose now that (b) holds and that the set of zeros of h is not finite. Then the Analytic Continuation Principle allows to assume the existence of a sequence $(z_n) \subset G$ tending to some boundary point t such that $h(z_n) = 0$ for all n . For each n , let us choose $r_n > 0$ so small that $B_n := B(z_n, r_n) \subset G$ and $\sup_{z \in B_n} \chi(t, B_n) \rightarrow 0$ ($n \rightarrow \infty$). Define the mappings $\tau_n(z) := r_n z + z_n$. Then $(\tau_n) \in N(t)$, whence there exists $f \in H(G)$ such that the sequence $\{h(\tau_n(z))f(\tau_n(z)) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{D})$, which is not possible, because $h(\tau_n(0))f(\tau_n(0)) = 0$ for all n . This contradiction shows that (b) implies (c). Finally, assume that the hypothesis of (c) is fulfilled, that is, $h(z) \neq 0$ for all $z \in G \setminus K$, for some compact set $K \subset G$. If $U \subset G \setminus K$ is an open ball then the operator $f \in H(G) \mapsto (h \cdot f)|_U \in H(U)$ has dense range by Runge’s theorem. Hence M_h has locally dense range near the boundary. Moreover, M_h is obviously locally stable near the boundary (for any h). An application of Theorem 4.5 yields that (c) implies (a). ■

Observe that from the last theorem we obtain further examples of linear strongly non-totally omnipresent operators: just take $G = \mathbb{D}$ and $T = M_h$, where h is the Blaschke product φ considered in Example 4.11.

5. DENSE LINEAR MANIFOLDS OF MONSTERS

The content of this section has been the main motivation for this paper. As indicated in Section 1, Luh [22] and Grosse-Erdmann [18] showed that, topologically speaking, the set of Luh-monsters is huge. We prove here that not only topologically but also algebraically Luh “created” too many monsters. The precise formulation for this statement will be given in Theorem 5.2. Nevertheless, a more general result can be stated.

THEOREM 5.1. *Assume that (S_j) is a countable family of linear totally omnipresent operators on $H(G)$. Then there exists a dense linear submanifold $M \subset H(G)$ such that $\mathcal{M}(S_j) \supset M \setminus \{0\}$ for all j .*

Proof. Fix a dense sequence $\{t_k : k \in \mathbb{N}\}$ in ∂G . For each $k \in \mathbb{N}$, fix a sequence $(\tau_n^{(k)}) \in N(t_k)$. By Proposition 2.2, the sequence $T_n^{(k,j)} : X \rightarrow Y$ ($n \in \mathbb{N}$) is densely hereditarily hypercyclic for every $k \in \mathbb{N}$ and every j , where

$X = Y := H(G)$ and $T_n^{(k,j)} := C_{\tau_n^{(k)}} S_j$. Then the hypotheses of Theorem 3.1 are fulfilled. Hence there is a dense linear manifold $M \subset H(G)$ with

$$M \setminus \{0\} \subset \bigcap_{k,j} HC((C_{\tau_n^{(k)}} S_j)).$$

But observe that the last intersection is included in $\mathcal{M}(S_j)$ for each j , because in order that $S_j f$ be a holomorphic monster it is sufficient to see its wild behaviour only near the points of a *dense* boundary subset, see [6, Lemma 2.1]. This drives us to $M \setminus \{0\} \subset \mathcal{M}(S_j)$ for all j . ■

The last theorem yields immediately the next corollary. Recall that a point $a \in G$ should be fixed in the definition of Luh-monster.

THEOREM 5.2. *Assume that $G \subset \mathbb{C}$ is a simply connected domain and that $a \in G$. Then there exists a dense linear submanifold of $H(G)$ whose non-zero members are Luh-monsters.*

Proof. Let us define the operators S_j ($j \in \mathbb{Z}$) as $S_j = D^j$ ($j \in \mathbb{N}_0$), $S_j = D_a^j$ ($-j \in \mathbb{N}$). Now just apply Theorem 5.1 and take in mind that

$$\{\text{Luh-monsters}\} = \bigcap_{j \in \mathbb{Z}} \mathcal{M}(S_j). \quad \blacksquare$$

Of course, Theorem 5.1 holds when the sequence (S_j) is changed by just a single operator T on $H(G)$. One can believe that the same assertion of Theorem 5.1 would hold just by assuming that T is a linear *strongly* omnipresent operator. This is false. As a matter of fact, it can happen that $\mathcal{M}(T)$ does *not* contain any manifolds of dimension *greater than one*. Indeed, consider the linear operator $Tf(z) = f(0) \cdot \varphi(z)$ on $H(\mathbb{D})$, where $\varphi \in H(\mathbb{D})$ is a fixed holomorphic monster (see [9, Example 2.9]). Assume that M is a linear manifold with $M \setminus \{0\} \subset \mathcal{M}(T)$ and $\dim(M) \geq 2$. Hence we can select two linearly independent functions f, g in M . Since $\mathcal{M}(T) = \{h \in H(\mathbb{D}) : h(0) \neq 0\}$ we have $\alpha := f(0) \neq 0 \neq g(0) =: \beta$. Define $h(z) := f(z) - \frac{\alpha}{\beta}g(z)$. By linear independence, $h \in M \setminus \{0\}$, whence $h \in \mathcal{M}(T)$. Thus, $0 \neq h(0) = \alpha - \alpha = 0$, which is a contradiction. This shows that $\dim(M) \leq 1$, as required. By the way, this shows that the operator T is not totally omnipresent.

6. RELATIONSHIP TO THE DI-OPERATORS. THE VOLTERRA OPERATOR

In this section we are considering briefly the boundary wild behaviour from another point of view. In 1995 Bernal-González proved [2] that for

every subset $A \subset G$ which is not relatively compact in G there exists a residual set M in $H(G)$ such that $f^{(n)}(A)$ is dense in \mathbb{C} for every $n \in \mathbb{N}_0$, and Calderón-Moreno showed later [12] that the same property is shared by certain kinds of infinite order differential and antidifferential operators and Volterra operators. This motivated us [7] to introduce the notion of dense-image operators or, briefly, *DI-operators*. A DI-operator is a (not necessarily linear) continuous self-mapping T on $H(G)$ satisfying that the set $M(T, A) := \{f \in H(G) : Tf(A) \text{ is dense in } \mathbb{C}\}$ is dense in $H(G)$ for every subset $A \subset G$ which is not relatively compact in G . From the fact that any of these subsets contains a sequence tending to some boundary point, it is easy to see that every totally omnipresent operator is a DI-operator. The converse is not true, see below. Let us summarize in one theorem several examples of relevant classes of DI-operators, see [7,12]. The interested reader should compare the following to the results of Section 4 and note that these earlier results improve in part the content of the next theorem.

THEOREM 6.1. *Let $G \subset \mathbb{C}$ be a domain, $\Phi(z), \Psi(z)$ two power series as in Theorem 4.1, and let $\varphi : G \times G \rightarrow \mathbb{C}$ be a holomorphic function with respect to both variables. Assume also that $\varphi_1 \in H(G, G), \varphi_2 \in H(\mathbb{C}), \varphi_3 \in H(G)$. We have:*

- (a) *If Φ is non-zero then $\Phi(D)$ is a DI-operator.*
- (b) *If G is simply connected, $a \in G$ and V_φ is the Volterra operator associated to a, φ , then V_φ is a DI-operator if and only if for every compact subset $L \subset G$ and every $A \subset G$ which is not relatively compact there exist $b \in A \setminus L$ and $s \in G$ such that $\varphi(b, s) \neq 0$. In addition, if φ satisfies this property then $\Phi(D) + V_\varphi$ is a DI-operator. In particular, if at least one of Φ, Ψ is non-zero then $\Phi(D) + \Psi(D_a^{-1})$ is a DI-operator.*
- (c) *Every onto linear operator is a DI-operator.*
- (d) *The operator C_{φ_1} is DI if and only if φ_1 is proper. In particular, if $G = \mathbb{C}$, then C_{φ_1} is a DI-operator if and only if φ_1 is a non-constant polynomial.*
- (e) *The operator L_{φ_2} is DI if and only if φ_2 is non-constant.*
- (f) *The operator M_{φ_3} is DI if and only if the set of zeros of φ_3 is finite.*

In [7] it is proved that DI-property implies omnipresence, and linear examples are exhibited showing that strong omnipresence (so omnipresence) does not imply DI-property. A non-linear example of a DI-operator which is not strongly omnipresent is also furnished, but we do not know whether every linear DI-operator is strongly omnipresent. If $G = \mathbb{C}$ and $\varphi(z) = z^{10}$ then by part (d) of the latter theorem the linear operator C_φ is DI (it is also strongly omnipresent, because φ is not constant), but it is not totally omnipresent, as we saw after Conjecture 4.12. Moreover, if $G = \mathbb{C}, a = 0$

and $\varphi(z, t) := \sin(\pi z)$ then the linear operator V_φ is strongly omnipresent by [8] but it is not DI (so not totally omnipresent): choose $L = \emptyset$ and $A = \mathbb{N}$ in part (b) of Theorem 6.1.

Now, we focus our attention on the Volterra operator of the first kind V_φ and conclude this paper with the statement of two results whose contents and proofs are analogous to the corresponding ones in [8]. Hence their proofs are left to the interested reader. In fact, similarly to [8], the first result below can be used to prove the second one as well as some assertions of Theorem 4.1. In the following, G is a simply connected domain of \mathbb{C} and a is a fixed point in G . In addition, if B is a closed ball in G , then $A(B)$ denotes the Banach space of all functions that are continuous in B and holomorphic in B^0 , endowed with the maximum norm $\|\cdot\|_B$. With the same norm we endow the subspace $A_b(B)$ consisting of all functions of $A(B)$ with a zero at b , where $b \in \partial B$.

THEOREM 6.2. *Let $S : H(G) \rightarrow H(G)$ be an operator and $\varphi : G \times G \rightarrow \mathbb{C}$ a holomorphic function with respect to both variables. Then the operator on $H(G)$ defined by*

$$Tf(z) = Sf(z) + V_\varphi(z)$$

is totally omnipresent if there exists a compact set $K \subset G$ such that for each closed ball $B' \subset G \setminus K$ there is a closed ball B with $B' \subset B \subset G \setminus K$ and a point $b \in \partial B$ such that

- (a) *the operator S extends continuously to a mapping*

$$\tilde{S} : A(B) \rightarrow A(B'),$$

- (b) *the mapping $\tilde{T} : A_b(B) \rightarrow A(B')$ defined by*

$$\tilde{T}f(z) = \tilde{S}f(z) + \int_b^z f(t)\varphi(z, t) dt \quad (z \in B')$$

has dense range.

THEOREM 6.3. *Assume that $\varphi : G \times G \rightarrow \mathbb{C}$ is holomorphic and that there exist $N \in \mathbb{N}_0$ and a compact set $K \subset G$ such that*

$$\frac{\partial^N \varphi}{\partial z^N}(w, w) \neq 0 = \frac{\partial^n \varphi}{\partial z^n}(w, w) \quad (n = 0, 1, \dots, N - 1) \quad \text{for all } w \in G \setminus K.$$

Then the operator V_φ is totally omnipresent.

We propose here the problem of characterizing the total omnipresence of V_φ in terms of φ .

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